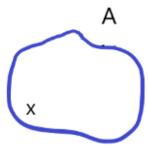
Dr. Ujjwal Kumar Pahari, Assistant Professor, Department of Mathematics

Netaji Nagar Day College

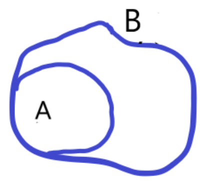
Topic for – Semester 2   Paper- MTMA CC2 (Real Analysis)

Basic Concepts of Real Analysis

Set: A well-defined collection of distinct objects is called a set. A set is usually denoted by capital letters A, B, X …. and an element of a set is denoted by small letters a, b, x, …When x is an element of a set A, it is expressed by the symbol $x \in A$. When x is not an element of a set A, it is expressed by the symbol $x \in A$.



Subsets: Let A and B two sets. If $x \in A \Rightarrow x \in B$, then A is said to be a subset of B, denoted by $A \subset B$ or $B \supset A$.This means that each element of A is an element of B. In this case B is said to be a superset of A. Whenever $A \subset B$ we say that B contains the set A.



Natural Numbers: The natural numbers are: 1, 2, 3, ….. The set of Natural numbers is denoted by **N.**

Whole Numbers: Whole numbers are: 0, 1, 2, 3,……. The set of whole numbers is denoted by **W**.

Integers: Integers are: …….. -3, -2, -1, 0, 1, 2, 3, …… The set of Integers is denoted by **Z**.

Positive integers are: 1, 2, 3, ………

Negative integers are: -1, -2, -3, ………..

The number **Zero '0' is an integer** but it is neither positive integer nor negative integer.

Rational numbers: A rational number is of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

Every integer is a rational number. The set of rational numbers is denoted by **Q.** $Z \subset Q$.

Between two rational numbers there exists infinite number of elements of Q. But there are some gaps between the rational numbers in form of irrational numbers.

Irrational numbers: The numbers which are not rational numbers are irrational numbers. Example: $\sqrt{5},\ \sqrt[3]{2},\ \log 2\,,\ \pi,\ e$ etc.

Real numbers: The set containing all rational as well as irrational numbers is called the set of real numbers, denoted by **R.**

$$N \subset W \subset Z \subset Q \subset R$$

■The set of rational numbers **Q** is an ordered field but not complete. Q is dense in R as well as dense-in-itself.

■ The set of Real numbers **R** is a complete ordered field. It is also dense-in-itself.

Cantor-Dedekind Axiom: There is a one-to-one correspondence between the set of all points on a line and the set of all real numbers.

Equivalent set: A set A is said to be equivalent to a set B (or equipotent or similar to B) if ∃ a bijection $f: A \to B$ and is denoted by $A \sim B$.

Enumerable set: Let S be a subset of R. S is said to be enumerable (or denumerable or countably infinite) if ∃ a bijection $f: N \to S$ i.e., if S and N are equipotent sets.

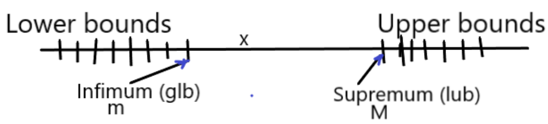
A set which **either finite or enumerable** is said to be **countable set**.

- The set N, Q, the set of rational numbers in [0, 1] are enumerable.
- R, (0, 1), [0, 1], the set of all irrational numbers in [0, 1] are non-denumerable. Also, the set of all irrational numbers are uncountable.

Concept of bounds: A set S ⊂ R of real numbers is **bounded above** if there exists a real number u ∈ R, called **an upper bound** of S, such that $x \le u$ for every x ∈ S.

Similarly, S is **bounded below** if there exists a real number l∈ R, called **a lower bound** of S, such that $x \ge l$ for every $x \in S$.

S is said to be **a bounded set** if S is bounded above as well as bounded below, i.e., ∃ real numbers l and u such that $l \le x \le u,\ \ \forall x \in R$.



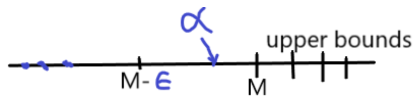
Example: Let $A = \{x \in R: 1 < x < 2\}$ and $B = \{x \in R:\ 1 \le x \le 2\}$. Both the sets are bounded above, 2 being an upper bound. Also, the numbers greater than 2 are also upper bounds.

Both the sets Both the sets are bounded below, 1 being a lower bound. Also, the numbers less than 1 are also lower bounds.

Both the sets are bounded sets. The null set ∅ is also an example of bounded set.

Supremum or least upper bound (lub): Let S be a subset of R. If S be bounded above then an upper bound of S is said to be the supremum of S if it is less than every other upper bound of S. That is, for a non-empty set S bounded above $\exists$ a real number M (called supremum of S) such that i) $x \leq M \quad \forall\ x \in S$

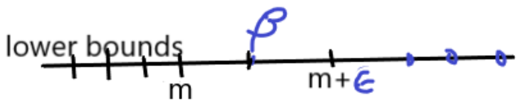
ii) $\forall \in (> 0)$ however small, $\exists\ \alpha \in S$ such that $M - \varepsilon < \alpha \leq M$.



Supremum of a set may or may not be a member of a set. In the above examples, the number 2 is the supremum of both the sets A and B but $2 \notin A$ and $2 \in B$.

Infimum or greatest lower bound (glb): Let S be a subset of R. If S be bounded below then a lower bound of S is said to be the infimum of S if it is greater than every other lower bound of S.). That is, for a non-empy set S bounded below $\exists$ a real number m (called infimum of S) such that i) $x \geq m \quad \forall\ x \in S$

ii) $\forall \in (> 0)$ however small, $\exists\ \beta \in S$ such that $m \leq \beta < m + \epsilon$.



Infimum of a set may or may not be a member of a set. In the above examples, the number 1 is the infimum of both the sets A and B but $1 \notin A$ and $2 \in B$.

Completeness Property (axiom of lub): Every non-empty subset of R that is bounded above has a least upper bound (or a supremum).

Theorem: A non-empty set S bounded below has its infimum (greatest lower bound.

Oscillation of a bounded set: If S be a bounded set with supremum M and infimum m, then $M - m$ is defined as oscillation of the set.

Example: If S be bounded and $|x| \leq A \ \ \forall\, x \in S$, then $\forall\, x, y \in S \ \ |x - y| \leq |x| + |y| \leq 2A$.

Archimedean Property of R:

If $x, y \in R$ and $x > 0, y > 0$, then $\exists$ a natural number n such that $ny > x$.

Deductions: i) If $x \in R$, then $\exists$ a natural number n such that $n > x$.

ii) If $x \in R$ and $x > 0$, then $\exists$ a natural number n such that $0 < \frac{1}{n} < x$.

iii) If $x \in R$ and $x > 0$, then $\exists$ a natural number m such that $m - 1 \leq x < m$.

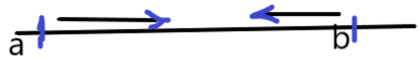
iv) If $x \in R$, then $\exists$ an integer m such that $m - 1 \leq x < m$.

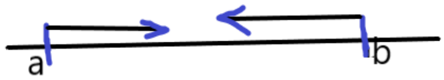
Note: $+\infty$ and $-\infty$ **are not real numbers.** The set R together with the two symbols $+\infty$ and $-\infty$ is called extended set of real numbers.

Linear point set: A set of real numbers (i.e., any subset of R) is defined as a linear point set.

Intervals: Let $a, b \in R$ and $a < b$.

Open interval $(a, b) = \{x \in R : a < x < b\}$





Closed interval $[a, b] = \{x \in R : a \leq x \leq b\}$
Semi closed interval $(a, b] = \{x \in R : a < x \leq b]$ or $[a, b) = \{x \in R : a \leq x < b\}$
$(a, \infty) = \{x \in R : a < x < \infty\}$
$[a, \infty) = \{a \leq x < \infty\}$

$(-\infty, b) = \{x \in R : -\infty < x < b\}$

$(-\infty, b] = \{x \in R : -\infty < x \leq b\}$

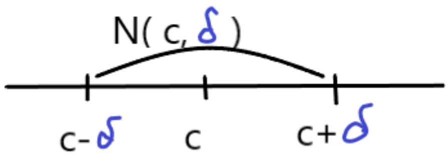
Neighbourhood: A subset $S \subset R$ is said to be a nbd of $c \in R$ if ∃ an open interval (a, b) such that $c \in (a, b) \subset S$.

■ An open bounded interval containing c is a nbd of c.

■ A closed bounded interval containing c may not be a nbd of c. $2 \in [2, 5]$ but the closed interval [2, 5] is not a nbd of 2 because we cannot find an open interval containing c which is a subset of [2, 5].

■ A non-empty finite set is not a nbd of any point.

■ $\boldsymbol{N(c, \delta)} = (c - \delta, c + \delta)$ is known δ- neighbourhood of $c \in R$ where $\delta > 0$.



$\boldsymbol{N'(c, \delta)} = (c - \delta, c + \delta) - \{c\}$ is known as deleted δ-neighbourhood of c where $\delta > 0$.

Limit point (accumulation point or cluster point): Let S be a subset of R. A point c in R (which may or may not be a member of S) is said to be a limit point of S if every nbd of c contains a point of S other than c. i.e., every deleted nbd of c contains a point of S. i.e., if

$\forall\, \varepsilon (> 0)\quad N'(c, \varepsilon) \cap S \neq \emptyset$.

Theorem: If c be a limit point of S then $\forall\, \varepsilon\, (> 0)$ $N(c, \varepsilon)$ contains an infinite number of members of S.

- **A set may have no limit point:** the set N of natural numbers has no limit point. Also, a finite set has no limit point.
- **A set may have only one limit point:** For the set $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, '0' is the limit point but '0' is not a member of the set S.
- **A set may have more than one limit points:** For the set $\{1,\ 1+1,\ \frac{1}{2},\ 1+\frac{1}{2},\ \frac{1}{3}, 1+\frac{1}{3},\ \ldots\ldots\}$ '0' and '1' are two limit points.
- **A set may have infinite number of limit points:** For the open interval (a, b) each point in the closed interval [a, b] is a limit point.

Derived set: Let S be subset of R. The set of all limit points of S is said to be the derived set of S and is denoted by S'. $N' = \emptyset,\quad Z' = \emptyset,\quad Q' = R,\quad R' = R,\quad \emptyset' = \emptyset,\quad (a,b)' = [a,b]$.

Result: A finite set has no limit point.

Proof: If possible, let *c* be a limit point of a finite set S. Then $\forall\ \varepsilon\ (>0)\quad N(c,\varepsilon)$ must contain an infinite number of members of S, which contradicts the hypothesis that S has a finite number of members.

Bolzano-Weierstrass theorem (on set): Every bounded infinite subset of R has a limit point (in R).

Few definitions

- **Interior point:** Let S be subset of R. A point $\alpha \in S$ is said to be an interior point of S if $\exists\ \varepsilon\ (>0)$ s. t. $N(\alpha,\varepsilon) \subset S$

  Remark: An interior point of a set must be a member of the set.
- **Boundary point:** Let S be subset of R. A point $\alpha \in R$ is said to be a boundary point of S if $\forall\ \varepsilon\ (>0)\ N(\alpha,\varepsilon)$ contains points of s as well as point not belonging to S.

  Remark: A boundary point of a set may or may not be a member of a set.
- **Exterior point:** Let S be subset of R. A point $\alpha \in R$ is said to be an exterior point of S if $\exists\ \varepsilon\ (>0)$ s. t. $N(\alpha,\varepsilon) \cap S = \emptyset$.

  Remark: An exterior point of S must not be a member of S.
- **Isolated point:** Let S be subset of R. A point $\alpha \in S$ is said to be an isolated point of S if $\exists\ \varepsilon\ (>0)$ s. t. $N(\alpha,\varepsilon) \cap S = \{\alpha\}$.

  i.e., α is an isolated point of S if $\exists\ \varepsilon\ (>0)\ \ s.t.\ \ N(\alpha,\varepsilon)$ contains no point of S other than α.

  Remark: An isolated point of a set must be a member of the set.

- **Adherent point:** Let S be subset of R. A point $\alpha \in R$ is said to be an adherent point of S if every nbd of α contains a point of S. i.e., if $\forall\ \varepsilon\ (>0)\ \ N(\alpha,\varepsilon)\cap S \neq \emptyset$.

  Remark: An adherent point may or may not belong to the set and it is either an isolated point or a limit point of the set.

Closure of a set: Let S be a subset of R. The set of all adherent points of S is said to be the closure of S and is denoted by $\bar{S}$. Thus, closure of S is defined to be the set of all points of S as well as the limit points of S. i.e., $\bar{S} = S \cup S'$.

- **Dense Set:** Let A and B be two subsets of R. If $A \subset B$ and every point of B is a limit point of A, i.e., $x \in B \Rightarrow x \in A', (or\ \ B \subset A')$, then A is said to be dense in B. For example, the set Q is dense in R, since $Q \subset R$ and every Real number is a limit point of Q (as $Q' = R$).

If $A = B$, we obtain the following definition:

If every point of a set S is a limit point of itself i.e., if $S \subset S'$, then S is said to be **dense-in-itself.** For example, the set of rational numbers Q is dense-in-itself. The set of real numbers R is dense-in-itself. Also, every open set is dense-in-itself.

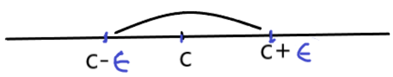
- **Perfect set:** Let S be a subset of R. S is said to be a perfect set if it is both closed and dense-in-itself.

  We know that a set S is closed if $S' \subset S$. Also, if $S \subset S'$, then it is dense-in-itself.

  Therefore, a set is perfect if $S = S'$.

  Any closed interval [a, b], the void set $\emptyset$, the set R are the examples of perfect sets.

Theorem: An **interior point** of a set $S \subset R$ is a **limit point** of S.



Proof: Let c be an interior point of S. If it is not a limit point of S, then $\exists\ \ \varepsilon\ (>0)\ \ s.t.$ $N'(c,\varepsilon)$ does not contain any element of S. Hence, no $(+)ve\ \ \delta$ can be found *s.t.* $N(c,\delta) \subset S$ which contradicts the fact that c is an interior point. Hence, c is a limit point of S.

Theorem: Any point α is either an interior point or a boundary point or an exterior point of a given linear point set S.

Proof: Let S be a given linear point set and α be any point. If α is not a boundary point of S, $\exists\ \varepsilon\,(>0)\ \ s.t.$ either $N(\alpha,\varepsilon)$ contains only points of S or $N(\alpha,\varepsilon)$ contains only points not of S.

In case-I, $N(\alpha,\varepsilon)\subset S$ i.e. α is an interior point of S.

In case-II, $N(\alpha,\varepsilon)\cap S=\emptyset$ in which case α is an exterior point of S.

Theorem: A boundary point of a set $S\subset R$ is either a limit point of S or an isolated point of S.

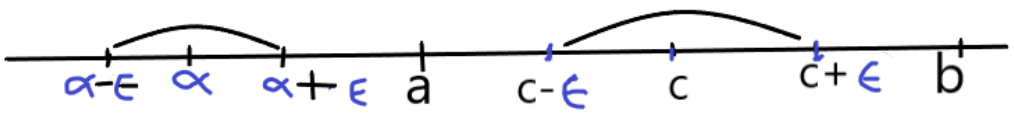
Proof: If c be a boundary point of S, then $\forall\ \varepsilon\,(>0)\ \ N(c,\varepsilon)$ contains points of S as well as points not of S. If $\forall\ \varepsilon\,(>0)\ \ N(c,\varepsilon)$ contains points of S other than c, then c is the limit point of S. If this does not hold $\exists\ \varepsilon\,(>0)\ \ s.t. N(c,\varepsilon)$ does not contain points of S other than c. In this case $c\in S$ and further c is an isolated point of S.

Interior, Exterior, Boundary of a set S: Let S be a subset of R. The set of all interior points of S is said to be the *interior* of S and is denoted by *int* S (or by S^0). The set of all exterior points of S is said to be the *exterior* of S and is denoted by *ext* S. The set of all boundary points of S is said to be the *boundary* of S and is denoted by $\delta S\ \ or\ \ Bd\ S$.

Example: Let $I\ =\ (a,b)$ be an open interval. Then, every member of I is an interior point of I. The points a and b are boundary points of I. Also, any point $\alpha<a$ and $\beta>$ b are exterior points of I.

Solution: Let $c\in I$. Choosing a $(+)ve\ \ \varepsilon<min\,(c-a,b-c)$, we have $N(c,\varepsilon)\subset I$.

$\therefore\ \ c$ is an interior point of I. Since c is an arbitrary point of I, every member of I is an interior point of I.



$2nd$ Part: The point 'a' is a boundary point of I, since $\forall\ \varepsilon\,(>0)\ \ N(a,\varepsilon)$ contains points of I as well as points not of I. Similarly, b is also a boundary point of I.

$\boldsymbol{3rd}$ **Part:** If $\alpha < a$, then choosing $(+)ve\ \varepsilon < a - \alpha$, we get $N(\alpha, \varepsilon) \cap I = \emptyset$.

$\therefore$ α is an exterior point of I. Similarly, if $\beta > b$, then β is an exterior point of I.

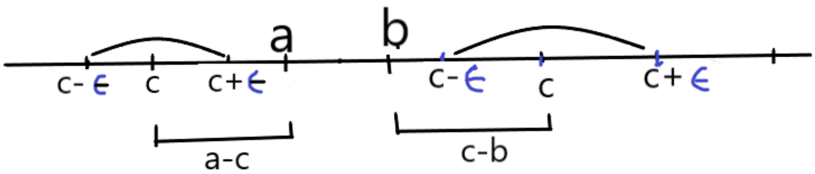
Note: Let $J = [a, b]$ be a closed interval. Then, every member x (a < x < b) of J is an interior point of I. The points a and b are boundary points of J. Also, any point $\alpha < a$ and $\beta > b$ are exterior points of J.

Open set and closed set: Let S be a subset of R. S is said to be an **open set** if each point of S is an interior point of S. S is said to be a **closed set** if no point outside of S is a limit point of S. i.e., if S contains all its limit points. In other words, if the derived set $S' \subset S$.

Example: An **open interval is an open set** and a **closed interval is a closed set**.

Solution: Let I = (a, b). Let $c \in I$. Then choosing $(+)ve\ \ \varepsilon < \min\,(c - a, b - c)$ it follows that $N(c, \varepsilon) \subset I$. So, c is an interior point of I. Thus, every point of I is an interior point of I. hence I is an open set.

Let J = [a, b]. Let $c \notin J$. If $c < a$, choosing a $(+)ve\ \ \varepsilon < a - c$, it can be seen that $N(c, \varepsilon)$ does not contain any element of J. Hence, c is not a limit point of J.



Again, if $c > b$, choosing a $(+)ve\ \ \varepsilon < c - b$, it follows that N (c, ε) does not contain any element of J. Hence, c is not a limit point of J. Thus, no point outside of J is a limit point of J. So, J is a closed set.

Example: A finite set is a closed set.

Solution: Since a finite set S contains only a finite number of points, it has no limit points. $\therefore$ $S' = \varphi$, Hence $S' \subset S$. $\therefore$ S is a closed set.

Theorem: The derived set S' of any set S is closed.

Theorem: Complement of an open set is closed and complement of a closed set is open.

- The sets R and $\emptyset$ are both open and closed sets.